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The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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From (77) we then get for stationary states*

$$E = -\frac{2\pi^2 me^4}{z^2h^2} + \frac{eHu}{4\pi mc}. \quad \ldots \quad \ldots \quad (90)$$

The same result can also be obtained by using the method adopted in Art. 7. This latter is to be preferred since it then becomes possible to determine whether $u < z$ or not.

15. Our hypothesis gives us that a quantum is emitted when the electron moves from a z-orbit to the $(z-1)$-orbit, or from a (z, u) orbit to the $(z-1, u-1)$ orbit. The number z may be termed the principal quantum number" and we postulate that *it always changes by unity* during a transition of the electron from one orbit to the next. Regarding the quantum number u, we assume the possibility of a switch from (z, u) to $(z-1, u+1)$ orbit; this would result from the condition

$$h\bar{\nu} - h\bar{\nu}' = E_1 - E_2 \quad \ldots \quad \ldots \quad (91)$$

We assume further that the transition from a (z, u) state to the $(z-1, u+k)$ state is possible, k being an integer, positive or negative. Our modified condition would, in that case, be

$$h\bar{\nu} + kh\bar{\nu}' = E_1 - E_2. \quad \ldots \quad \ldots \quad (92)$$

As $u_1 - u_2$ will here be equal to k, we shall get,

$$h\,[\,f(z_1) - f(z_2)\,] + h\,[\,\phi(u_1) - \phi(u_2)\,] = E_1 - E_2 \quad \ldots \quad (93)$$

leading to

$$h\,f(z) + h\phi(u) = E \quad \ldots \quad \ldots \quad (94)$$

and succeeding partial differential equations as before. The subsequent calculation of the energy-expressions will, therefore, retain their forms as can be easily verified in any of the cases considered before. It is

* Birtwistle: *Quantum Theory of the Atom*, p. 116.

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necessary that the quantity $E_1 - E_2$ should remain positive in order that an emission may occur; this entails corresponding limitations on the left hand side of (92). Since each stationary orbit has according to our scheme, a definite z and a definite u number, it is clear that the transit corresponding to $k = 0$, is to be treated as a particular case of the above, and the result thereof is to be deduced from the general formulæ by putting $k = 0$.

16. Let us now consider how a transition from a z orbit to the $(z-2)$ orbit can take place and what will be the consequence thereof. To be definite, we shall consider the case of single quantum orbits. According to our principle this can only happen in two stages, *viz.*, a transition from z to $(z-1)$ orbit and then from $(z-1)$ to the $(z-2)$ orbit. These two transitions may occur almost simultaneously, the second switch following the first very rapidly. In consequence two quanta $h\nu_1$ and $h\nu_2$ are liberated; these quanta may combine to form a new quantum $(h\nu_1 + h\nu_2)$ whose frequency is obviously the sum of the frequencies of the other two. We assume that such combinations of quanta do take place under suitably favourable circumstances. Thus in the transit of the electron from z to $(z-2)$ orbit we may get emissions of radiation of frequencies ν_1 and ν_2 or a single radiation of frequency $\nu_1 + \nu_2$. This can easily be extended to the general case where transit occurs from a z to a $(z-\gamma)$ orbit or from a (z, u) to a $(z-\gamma, u+k)$ orbit.

Thus we have from (29),

$$h\nu_1 = \frac{2\pi^2 e^4 m}{h^2}\left[\frac{1}{(z-1)^2} - \frac{1}{z^2}\right],$$

and
$$h\nu_2 = \frac{2\pi^2 e^4 m}{h^2}\left[\frac{1}{(z-2)^2} - \frac{1}{(z-1)^2}\right].$$

$$\therefore \quad h(\nu_1 + \nu_2) = \frac{2\pi^2 e^4 m}{h^2}\left[\frac{1}{(z-2)^2} - \frac{1}{z^2}\right] \qquad \ldots \quad (95)$$

For the general case we have

$$h\nu_1 = E(z) - E(z-1);$$

$$h\nu_2 = E(z-1) - E(z-2); \text{ etc.}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$h\nu_r = E(z-r+1) - E(z-r).$$

$$\therefore \qquad h(\nu_1 + \nu_2 + \ldots + \nu_r) = E(z) - E(z-r) \qquad \ldots \quad (96)$$

The principle of quantum combination strengthens our hypothesis whose mathematical form is enunciated in (42), (91) and (92). These considerations and the results like (58), (86), etc. easily lead to the rules of selection in the various cases.

[In this connection it may be of interest to note that in a paper published in the *Philosophical Magazine* No. 72, April 1931, Dr. H. L. Brose and Mr. E. H. Saayman have obtained $(n+\alpha)h\nu$ as the energy-expression for an harmonic oscillator, n being an integer and $0 < \alpha < 1$, on the basis of Bohr's requency condition and the definition of matrix multiplication: they have further expressed the hope of extending the method to the rotator].

ON A PARTICULAR TYPE OF SCROLLS.

BY C. N. SRINIVASIENGAR,
Central College, Bangalore.

1, The highest order of multiplicity that is possible for a multiple curve on a surface of degree n is $n-1$, and in this case the multiple curve is a straight line and the surface is ruled. Taking the multiple curve as the z-axis, the equation of the surface assumes the form

$$zu_{n-1} = v_{n-1} + w_n \qquad \ldots \quad (A)$$

where u_{n-1}, v_{n-1}, w_n represent homogeneous functions of x and y of degrees indicated by their suffixes. The generators of the surface (A) are given by

$$\left.\begin{aligned} y &= mx \\ z &= f_1(m) + xf_2(m) \end{aligned}\right\} \qquad \ldots \quad (B)$$

where $$f_1(m) = \frac{b_0 + b_1 m + \ldots + b_{n-1} m^{n-1}}{a_0 + a_1 m + \ldots + a_{n-1} m^{n-1}};$$

and $$f_2(m) = \frac{c_0 + c_1 m + \ldots + c_n m^{n-1}}{a_0 + a_1 m + \ldots + a_{n-1} m^{n-1}}.$$

Taking u_{n-1} and v_{n-1} in their most general forms, *i.e.* so as to involve all the terms, and such that there exists no common factor between them, we write down the following properties of the surface). The case $n = 3$ gives the well-known first type of the cubic scroll, several of whose properties have been studied by Cayley* and Salmon.

2. (1) *The points where the torsal generators*† *meet the multiple*

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* Cayley: *Collected Papers*, Vol. V, pp. 210—13.

Salmon: *Analytical Geometry of Three Dimensions*. Vol. II, Chapter XV.

† A torsal generator may be defined by any one of the following properties which are equivalent:—

(i) A generator which meets its consecutive generator.

curve are pinch-points. and the number of either on the surface is $2(n-2)$*

For, if the generator 'm' be torsal, $f'_1(m) = 0$, and the z-co-ordinate of the point where it meets the multiple line can easily be seen to be given as a root of the equation obtained as the m-discriminant of

$$z = \frac{b_0 + b_1 m + \dots + b_{n-1} m^{n-1}}{a_0 + a_1 m + \dots + a_{n-1} m^{n-1}}.$$

Next, if $(0, 0, \gamma)$ be a pinch-point, the equation $\gamma u_{n-1} - v_{n-1} = 0$ admits of the square of a linear factor. γ is therefore given by the m-discriminant of the same equation.

(2) ***The distance measured from any fixed point on the multiple line to its intersection with the generator whose parameter is m is a maximum or a minimum for the torsal generators.***

This property coupled with the number of torsal generators might help us in forming an idea of the shape of the surface.

(3) ***The surface possesses n generators parallel to one another.***

For the equations of the $n-1$ generators given by the values of m satisfying $a_0 + a_1 m + \dots + a_{n-1} m^{n-1} = 0$ are of the form $x =$ constant ; $y =$ constant, *i e*., these generators are parallel to the double line

---

(ii) A generator at all points along which the surface possesses the sama tangent plane.

(iii) A locus of parabolic points on the scroll.

(iv) An envelope of the non-rectilinear asymptotic lines of the scroll.

(v) A rectilinear line of curvature on the scroll (*vide* Q. 1551, J. I. M.S.)

Some writers use the term 'singular generator' in place of torsal generator. I would, however, use the term singular generator in a sense analogus to the use of the term 'Singular' in Differential Equations, *viz* , a generator not included in the general system of generators.

* The property (1) has been obtained for the quartic scroll with a triple line in Basset's *Geometry of Surfaces*, § 326 from entirely different considerations.

(4) ***The equations of the second system of asymptotic lines may be expressed in a finite form.***

To prove this, consider the representations of the surface by curvilinear co-ordinates.

$$\left.\begin{aligned} x &= p \\ y &= pq \\ z &= f_1(q) + pf_2(q) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (C)$$

The fundamental magnitudes of the second order are given by, (omitting the factor $\sqrt{EG - F^2}$)

$$L = 0\,; \quad M = -f_1'\,; \quad N = pf_1'' + p^2 f_2''.$$

The equation of the second systems of asymptotic lines reduces to

$$2f_1' \frac{dp}{dq} = pf_1'' + p^2 f_2'',$$

which is a differential equation of Bernoulli's type. The solution is

$$f_1' + pf_2' = Cp$$

The second system of asymptotic lines is therefore included among the curves of intersection of the scroll with the surfaces

$$f_1'\left(\frac{y}{x}\right) + xf_2'\left(\frac{y}{x}\right) = Cx.$$

(5) ***The Hessian of the surface is composed of the 2 (n — 2) planes, each containing the multiple line and a torsal generator, each plane being counted twioe.***

This is proved by actually writing down the equation of the Hessian of (A) It easily follows that the planes constituting the Hessian are the pinching planes at the pinch-points. In particular, for the cubic scroll, the two unodal planes repeated twice constitute the entire Hessian.*

* Question 1536 (1), *J. I. M. S.* I have since found that this result was known to Cayley.

(6) *On each torsal generator, there exists a definite point the indicatrix at which is a plane-cubic.*

I have elsewhere enunciated the result that if at a point on a surface, L, M, N all vanish, the point is either a double point or otherwise the indicatrix thereat is a cubic or higher curve. This gives the above result. The theorem is true in general for any ruled surface, for taking the equations of the ruled surface in the general form

$$x = \phi_1(p) + q\,\psi_1(p)\,;\; y = \phi_2(p) + q\,\psi_2(p)\,;\; z = \phi_3(p) + q\,\psi_3(p)$$

we have N = 0, and M = a function of p only. The equations L = 0, M = 0 thus give a number of points equal to the nnmber of torsal generators.

We can arrive at property (6) otherwise also in the case, as in the first type of the cubic scroll, when there exists on the surface a rectilinear directrix which is not a double line. Consider the point where this directrix meets a torsal generator. There are thus two different inflexional tangents at this point which is situate on a parabolic curve. The point not being a double point must have a cubical indicatrix.

This fact may be utilised to find the equation of the simple directrix line of the ruled cubic.

$$z(ux^2 + 2vxy + wy^2) = (ax^2 + 2hxy + by^2) + (a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3)$$

For the co-ordinates of the two points of its intersection with the torsal generators are obtained from the equations (C) coupled with M = 0, N = 0.

Also, ***if a ruled surface possesses two directrix lines, either one of these must be a double line or otherwise, the surface possesses no torsal generators.*** For the two directrix lines are necessarily non-coplanar. If a torsal generator exist meeting these lines, then the plane formed by the torsal generator and *either* of the directrices must touch the surface at every point along the torsal generator.

3. The number of torsal generators, pinch-points, etc, obtained for a surface of the form (A) undergoes alterations when the equation takes special forms. We might classify the different kinds of scrolls of the nth degree with a multiple line of order $n-1$ as follows:

Type I. There is no plane which forms a tangent plane (*i.e.* a locus of lines meeting the surface in three consecutive points at the double point) at all points of the multiple line.

Type II. There is one such plane.

... ...

Type $(n-1)$th. There are $(n-2)$ such planes. We can have sub-types when two or more of these planes coincide. In general, *if* $P=0$ *is a plane which touches the surface all along the multiple line,* P^2 *occurs as a factor of the Hessian.*

By way of illustration, we shall consider the quartic scroll with a triple line and form the following table :—

Type of Scroll.	Number of torsal generators.	Pinch-points.*	Parallel generators.
Type I. No common tangent plane along the triple line,	4	4	4
Type II. One common tangent plane throughout the triple line.	2	3	3
Type III. (*a*) Two common tangent planes	0	2	2
(*b*) Two coincident common tangent planes.	0	1	2

* *Vide* Salmon *loc. cit.* Vol. II. Chapter XVI. The types I consider here include Salmon's Types I and II. In the last typea III (*b*), the number of pinch-points is strictly speaking infinite. Salmon evidently means by the single pinch-point the point where all the three tangent planes coincide.

ON MATHIEU FUNCTIONS.

By Ram Shanker Varma, M.Sc.

In a recent paper,* Prof. Whittaker has shown that Mathieu functions satisfy the following recurrence formulae :

$$i\, \omega_n(x)\, ce_n'(ix) + \sigma_n(x)\, ce_n(ix) = \mu_n\, ce_{n-1}(ix) \quad \ldots \quad (1)$$

and

$$i\, \rho_n(x)\, ce_n'(ix) + \tau_n(x)\, ce_n(ix) = \nu_n\, ce_{n+1}(ix). \quad \ldots \quad (2)$$

where μ_n and ν_n are constants, and where

$$\omega_n(x) = \frac{\sinh x}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \eta\, ce_n\, \eta\, ce_{n-1}\, \eta}{\cosh^2 x - \cos^2 \eta}\, d\eta,$$

$$\sigma_n(x) = -\frac{\cosh x}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \eta\, ce_n'\, \eta\, ce_{n-1}\, \eta}{\cosh^2 x - \cos^2 \eta}\, d\eta,$$

$$\rho_n(x) = \frac{\sinh x}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \eta\, ce_n\, \eta\, ce_{n+1}\, \eta}{\cosh^2 x - \cos^2 \eta}\, d\eta,$$

$$\tau_n(x) = -\frac{\cosh x}{2\pi} \int_{-\pi}^{\pi} \frac{\sin \eta\, ce'_n\, \eta\, ce_{n+1}\, \eta}{\cosh^2 x - \cos^2 \eta}\, d\eta.$$

The object of the present paper is to develop a systematic theory of indefinite integrals involving Mathieu functions by the help of (1) and (2). An investigation† similar to this was made by the author for the parabolic cylinder functions ; but on account of the frequent occurrence of Mathieu functions in physical investigations, it seems worth while to place on record the theory of indefinite integrals involving Mathieu functions also.

All the results of this paper are believed to be new.

Equations (1) and (2) can be written

$$i\, ce_n'(ix) = S_n(x)\, ce_{n-1}(ix) - Q_n(x)\, ce_n(ix) \quad \ldots \quad (3)$$

and

$$i\, ce_n'(ix) = R_n(x)\, ce_{n+1}(ix) - P_n(x)\, ce_n(ix), \quad \ldots \quad (4)$$

---

* Whittaker, "On the Recurrence formulae of the Mathieu Functions," *J.L.M.S.*, Vol. IV. pp. 88–96.

† *J.I.M.S.*, Vol. XVIII (1930), pp. 226—232.

respectively, where

$$P_n(x) = \frac{\tau_n(x)}{\rho_n(x)}, \quad Q_n(x) = \frac{\sigma_n(x)}{\omega_n(x)},$$

$$R_n(x) = \frac{\nu_n}{\rho_n(x)}, \quad S_n(x) = \frac{\mu_n}{\omega(x)}.$$

Now let us suppose that

$$\int f(x)\, ce_n(ix)\, dx = A(x)\, ce_n(ix) + B(x)\, ce_{n+1}(ix),$$

where $A(x)$ and $B(x)$ are to be determined.

Differentiating and making use of the formulae (3) and (4), we get

$$f(x)\, ce_n(ix) = \{ A'(x) - A(x) P_n(x) + B(x) S_{n+1}(x) \}\, ce_n(ix)$$
$$+ \{ B'(x) + A(x) R_n(x) - B(x) Q_{n+1}(x) \}\, ce_{n+1}(ix).$$

In order that $A(x)$ and $B(x)$ be independent of ce's, we take

$$f(x) = A'(x) - A(x) P_n(x) + B(x) S_{n+1}(x)$$
$$0 = B'(x) + A(x) R_n(x) - B(x) Q_{n+1}(x). \quad \ldots \quad (5)$$

Hence it follows that

$$\int \{ A'(x) - A(x) P_n(x) + B(x) S_{n+1}(x) \}\, ce_n(ix)\, dx$$
$$= A(x)\, ce_n(ix) + B(x)\, ce_{n+1}(ix),$$

where $A(x)$ and $B(x)$ are connected by (5).

Next let us consider the integral

$$\int f(x)\, ce_m(i\phi)\, ce_n(i\psi)\, dx,$$

which we may suppose to be of the form

$$A(x)\, ce_m(i\phi)\, ce_n(i\psi) + B(x)\, ce_{m+1}(i\phi)\, ce_n(i\psi)$$
$$+ C(x)\, ce_m(i\phi)\, ce_{n+1}(i\psi) + D(x)\, ce_{m+1}(i\phi)\, ce_{n+1}(i\psi), \quad (6)$$

where ϕ and ψ are functions of x.

On differentiating, we obtain, in the manner indicated above, the following equations for the determination of A, B, C and D.

$$f(x) = A'(x) - A(x)\ \{\psi' P_n(\psi) + \phi' P_m(\phi)\},$$
$$+ B(x)\,\phi' S_{m+1}(\phi) + C(x)\,\psi' S_{n+1}(\psi)$$

$$0 = B'(x) + A(x)\,\phi' R_m(\phi) - B(x)\ \{\psi' P_n(\psi)$$
$$+ \phi' Q_{m+1}(\phi)\} + D(x)\,\psi' S_{n+1}(\psi),$$

$$0 = C'(x) + A(x)\,\psi' R_n(\psi) - C(x)\ \{\psi' Q_{n+1}(\psi)$$
$$+ \phi' P_m(\phi)\} + D(x)\,\phi' S_{m+1}(\phi),$$

$$0 = D'(x) + B(x)\,\psi' R_n(\psi) + C(x)\,\phi' R_m(\phi)$$
$$- D_2(x)\ \{\psi' Q_{n+1}(\psi) + \phi' Q_{m+1}(\phi)\}.$$

These equations lead us to

$$-\psi' R_n(\psi)\,B(x) = \tfrac{1}{2}D'(x) - D(x)\,\psi' Q_{n+1}(\psi) + h,$$

$$-\phi' R_m(\phi)\,C(x) = \tfrac{1}{2}D'(x) - D(x)\,\phi' Q_{m+1}(\phi) - h,$$

$$2\,\psi' R_n(\psi)\,\phi' R_m(\phi)\,A(x) = D'' - M(x)\,D' - N(x)\,D - O(x)\,h,$$

and $f(x) = A'(x) - A(x)\ \{\psi' P_n(\psi) + \phi' P_m(\phi)\}$

$$-\frac{D'}{2}\left\{\frac{\phi' S_{m+1}(\phi)}{\psi' R_n(\psi)} + \frac{\psi' S_{n+1}(\psi)}{\phi' R_m(\phi)}\right\}$$

$$+ D\left\{\frac{\phi' S_{m+1}(\phi)\,Q_{n+1}(\psi)}{R_n(\psi)} + \frac{\psi' S_{n+1}(\psi)\,Q_{m+1}(\phi)}{R_m(\phi)}\right\}$$

$$- h\left\{\frac{\phi' S_{m+1}(\phi)}{\psi' R_n(\psi)} - \frac{\psi' S_{n+1}(\psi)}{\phi' R_m(\phi)}\right\},$$

where

$$2\frac{dh}{dx} - h\left[\left(\frac{\psi''}{\psi'} + \frac{\phi''}{\phi'}\right) + \frac{d}{dx}\log R_n(\psi)\,R_m(\psi)\right.$$
$$\left. + \psi'\left\{P_n(\psi) + Q_{n+1}(\psi)\right\} + \phi'\left\{P_m(\phi) + Q_{m+1}(\phi)\right\}\right]$$

$$= D'\left[\frac{1}{2}\left(\frac{\psi''}{\psi'} - \frac{\phi''}{\phi'}\right) + \frac{1}{2}\frac{d}{dx}\log\frac{R_n(\psi)}{R_m(\phi)}\right.$$
$$\left. + \frac{1}{2}\psi'\left\{P_n(\psi) + Q_{n+1}(\psi)\right\} - \frac{1}{2}\phi'\left\{P_m(\phi) + Q_{m+1}(\phi)\right\}\right]$$

$$+ D \Big[\psi'^2 \Big\{ Q'_{n+1}(\psi) - Q_{n+1}(\psi) P_n(\psi) + R_n(\psi) S_{n+1}(\psi) \Big\}$$

$$- \phi'^2 \Big\{ Q'_{m+1}(\phi) - Q_{m+1}(\phi) P_m(\phi) + R_m(\phi) S_{m+1}(\phi) \Big\}$$

$$- \psi' Q_{n+1}(\psi) \frac{d}{dx} \log R_n(\psi) + \phi' Q_{m+1}(\phi) \frac{d}{dx} \log R_m(\phi) \Big],$$

$$M(x) = \frac{1}{2} \left(\frac{\psi''}{\psi'} + \frac{\phi''}{\phi'} \right) + \frac{1}{2} \frac{d}{dx} \log R_n(\psi) R_m(\phi)$$

$$+ \frac{3}{2} \Big\{ \psi' Q_{n+1}(\psi) + \phi' Q_{m+1}(\phi) \Big\} + \frac{1}{2} \Big\{ \psi' P_n(\psi) + \phi' P_m(\phi) \Big\};$$

$$N(x) = \psi'^2 \Big\{ Q'_{n+1}(\psi) - Q_{n+1}(\psi) P_n(\psi) + R_n(\psi) S_{n+1}(\psi) \Big\}$$

$$+ \phi'^2 \Big\{ Q'_{m+1}(\phi) - Q_{m+1}(\phi) P_m(\phi) + R_m(\phi) S_{m+1}(\phi) \Big\}$$

$$- 2 \psi' \phi' Q_{n+1}(\psi) Q_{m+1}(\phi) - \phi' Q_{m+1}(\phi) \frac{d}{dx} \log R_m(\phi)$$

$$- \psi' Q_{n+1}(\psi) \frac{d}{dx} \log R_n(\psi),$$

and

$$O(x) = \left(\frac{\psi''}{\psi'} - \frac{\phi''}{\phi'} \right) + \frac{d}{dx} \log \frac{R_n(\psi)}{R_m(\phi)}$$

$$+ \psi' P_n(\psi) - \phi' P_m(\phi) - \Big\{ \psi' Q_{n+1}(\psi) - Q_{m+1}(\phi) \Big\}.$$

We notice that the functions A, B, C and f have all been expressed in terms of the function D and hence particular cases of the general integral (6) can be easily worked out by assigning various values to D. One such simple case is given by supposing that D = 0. We have, then,

$$\int \Big[A''(x) - A(x) \{ \psi' P_n(\psi) + \phi' P_m(\phi) \}$$

$$- h \left\{ \frac{\phi' S_{m+1}(\phi)}{\psi' R_n(\psi)} - \frac{\psi' S_{n+1}(\psi)}{\phi' R_m(\phi)} \right\} \Big] ce_m(i\phi)\, ce_n(i\psi)\, dx$$

$$= A(x)\, ce_m(i\phi)\, ce_n(i\psi) - \frac{h}{\psi' R_n(\psi)} ce_{m+1}(i\phi)\, ce_n(i\psi)$$

$$+ \frac{h}{\phi' R_m(\phi)} ce_m(i\phi)\, ce_{n+1}(i\psi), \qquad (7)$$

where

$$2\,\psi'\,R_n(\psi)\,\phi'\,R_m(\phi)\,A(x) = -h\Big[\left(\frac{\psi''}{\psi'} - \frac{\phi''}{\phi'}\right)$$

$$+\frac{d}{dx}\log\frac{R_n(\psi)}{R_m(\phi)} + \psi'\,P_n(\psi) - \phi'\,P_m(\phi)$$

$$-\left\{\psi'\,Q_{n+1}(\psi) - \phi'\,Q_{m+1}(\phi)\right\}\Big],$$

and

$$2\frac{dh}{dx} = h\Big[\left(\frac{\psi''}{\psi'} + \frac{\phi''}{\phi'}\right) + \frac{d}{dx}\log R_n(\psi)\,R_m(\phi)$$

$$+\psi'\left\{P_n(\psi) + Q_{n+1}(\psi)\right\} + \phi'\left\{P_m(\phi) + Q_{m+1}(\phi)\right\}\Big].$$

If we now connect ψ and ϕ by the relation

$$A'(x) = A(x)\left\{\psi'\,P_n(\psi) + \phi'\,P_m(\phi)\right\}$$

$$+\,h\left\{\frac{\phi'\,S_{m+1}(\phi)}{\psi'\,R_n(\psi)} - \frac{\psi'\,S_{n+1}(\psi)}{\phi'\,R_m(\phi)}\right\},$$

we deduce from (7) that

$$A(x)\,ce_m(i\phi)\,ce_n(i\psi) - \frac{h}{\psi'\,R_n(\psi)}\,ce_{m+1}(i\phi)\,ce_n(i\psi)$$

$$+\frac{h}{\phi'\,R_m(\phi)}\,ce_m(i\phi)\,ce_{n+1}(i\psi) = \text{constant}.$$

WEIRSTRASS' NON-DIFFERENTIABLE FUNCTION.

K. VENKATACHALA IYENGAR.

The best condition that has been given hitherto for the differentiability of $f(x) = \sum_0^\infty a^n \cos b^n \pi x$ is that $ab > 1 + \frac{3\pi}{2}(1 - a)$, where $0 < a < 1$ and b an odd integer—a condition due to Bromwich. The case when b is even has been mentioned by Hardy in the T.A.M.S. (1916) in which he gives a condition given by Dini. In this paper better conditions are found for these cases as well as for the case of Cellerier's function, *viz.*, $f(x) = \sum_0^\infty a^n \sin b^n \pi x$.

(i) ***Let b be odd.***

$f(x) = \sum_0^\infty a^n \cos b^n \pi x$. It is obvious that it is sufficient if we consider only positive values of x. It is shown here that we can find sequences $h_1{}^r$ and $h_2{}^r$ which tend to zero when $r \to \infty$ and for which

$$\operatorname*{Lt}_{r \to \infty} \frac{f(x + h_1{}^r) - f(x + h_2{}^r)}{h_1{}^r - h_2{}^r} = \infty$$

and that the limit becomes equal to $-\infty$ for two different sequences. Moreover, it is proved here that under the conditions obtained, the right-hand and left-hand derivates do not exist.

Now $b^r x$ always lies between two even integers so that

$$b^r x = k_r - \varepsilon_r$$

where k_r is an even integer and $0 \leqslant \varepsilon_r \leqslant 2$. Take $h_1{}^r = \frac{\varepsilon_r}{b^r}$ and $h_2{}^r = \frac{1 + \varepsilon_r}{b^r}$. It is obvious that $h_1{}^r$ and $h_2{}^r \to 0$ as $r \to \infty$.

Then

$$\frac{f(x + h_1{}^r) - f(x + h_2{}^r)}{h_1{}^r - h_2{}^r} = \sum_0^{r-1} \frac{a^n [\cos b^n \pi (x + h_1{}^r) - \cos b^n \pi (x + h_2{}^r)]}{h_1{}^r - h_2{}^r}$$

$$+ \sum_r^\infty \frac{a^n [\cos b^n \pi (x + h_1{}^r) - \cos b^n \pi (x + h_2{}^r)]}{h_1{}^r - h_2{}^r},$$

$$= \phi_1(x) + \phi_2(x).$$

Now it is easy to shew by means of the first mean value theorem of the Differential Calculus that

$$|\phi_1(x)| < \pi \sum_{0}^{r-1} a^n b^n < \frac{\pi . a^r b^r}{ab - 1}.$$

(Here it is assumed that $ab > 1$).

Now $\cos b^n \pi (x + h_1{}^r) = \cos b^{n-r} \pi k_r = +1$ if $n \geqslant r$,

and $\cos b^n \pi (x + h_2{}^r) = \cos b^{n-r} \pi (k_r + 1) = -1$ if $n \geqslant r$.

Hence $\phi_2(x) = -2a^r b^r (1 + a + a^2 + \ldots) = -\frac{2a^r b^r}{1-a}$.

Therefore $$\operatorname*{Lt}_{r \to \infty} \frac{f(x + h_1{}^r) - f(x + h_2{}^r)}{h_1{}^r - h_2{}^r} = -\infty,$$

provided that $\frac{2a^r b^r}{1-a} > \frac{a^r b^r . \pi}{ab - 1}$, that is, if $ab > 1 + \frac{\pi}{2}(1 - a)$

since $a^r b^r \to \infty$ as $r \to \infty$, ab being > 1.

Now $b^r x$ also lies between two odd integers, and we have $b_r x = k_r - \varepsilon$ where k_r is odd and $0 \leqslant \varepsilon_r \leqslant 2$, and choosing $h_1{}^r$ and $h_2{}^r$ in a similar manner, we have

$$\cos b^n \pi (x + h_1{}^r) = \cos b^{n-r} k_r \pi = -1 \text{ if } n \geqslant r,$$

and $$\cos b^n \pi (x + h_2{}^r) = \cos b^{n-r} (k_r + 1) \pi = +1 \text{ if } n \geqslant r.$$

Hence, in this case

$$\operatorname*{Lt}_{r \to \infty} \frac{f(x + h_1{}^r) - f(x + h_2)}{h_1{}^r - h_2{}^r} = +\infty,$$

provided that $\frac{2a^r b^r}{1-a} > \frac{a^r b^r \pi}{ab-1}$ or $ab > 1 + \frac{\pi}{2}(1 - a)$.

Hence this condition suffices for the non-existence of the right-hand derivate at any point. By writing $b^r x = k_r + \varepsilon_r$ where $0 \leqslant \varepsilon_r \leqslant 2$ and choosing k_r to be alternately odd or even we can, in a similar manner shew that the left-hand derivate also does not exist.

(ii) *The case when b is even.*

In this case h_1^r and h_2^r can be chosen as follows:

Let $b^r x = k_r - \varepsilon_r$ where k_r is an odd integer and $0 \leqslant \varepsilon_r \leqslant 2$,

and
$$h_1^r = \frac{\varepsilon_r}{b^r}, \quad h_2^r = \frac{\varepsilon_r + 1}{b^r}.$$

Then $\cos b^n \pi (x + h_1^r) = -1$, $\cos b^n \pi (x + h_2^r) = +1$ and if $n > r$,
$$\cos b^n \pi (x + h_1^r) = +1, \cos b^n \pi (x + h_2^r) = +1;$$
proceeding similarly we get
$$\phi_2(x) = 2a^r b^r.$$

Hence
$$\underset{r \to \infty}{\text{Lt}} \frac{f(x + h_1^r) - f(x + h_2^r)}{h_1^r - h_2^r} = +\infty$$

if $2a\,b^r > \dfrac{a^r b^r \pi}{ab - 1}$ or if $ab > 1 + \dfrac{\pi}{2}$.

If we choose h_1^r and h_2^r in the following manner, *viz.*, $b^r x = k_r - \varepsilon_r$, where k_r is even and $0 \leqslant \varepsilon_r \leqslant 2$, we can prove in a similar manner that under the same condition

$$\underset{r \to \infty}{\text{Lt}} \frac{f(x + h_1^r) - f(x + h_2^r)}{h_1^r - h_2^r} = -\infty.$$

Hence the condition suffices for the non-existence of the right-hand derivate.

Now if we take k_r to be an odd integer in one case and even in the other case where

$$b_r x = k_r + \varepsilon_r, \quad h_1^r = \frac{-\varepsilon_r}{b^r}, \quad h_2^r = \frac{-1 - \varepsilon_r}{b^r}$$

we can shew that $\underset{r \to \infty}{\text{Lt}} \dfrac{f(x + h_1^r) - f(x + h_2)}{h_1^r - h_2^r}$ oscillates between $+\infty$ and $-\infty$. This shows that the left hand derivate also does not exist.

Cellerier's Function.

$$f(x) = \sum_0^\infty a^n \sin b^n \pi x.$$

(i) Consider the case when $b = (4m + 1)$ when m is an integer. Let $b_r x = k_r - \varepsilon_r$ where $k_r = (2p + \frac{1}{2})$ where p is an integer and $0 \leqslant \varepsilon_r \leqslant 2$.

Take $$h_1{}^r = \frac{\varepsilon_r}{b_r}, \quad h_2{}^r = \frac{\varepsilon_r + \frac{1}{2}}{b^r}.$$

Then it is obvious that $h_1{}^r$ and $h_2{}^r$ both tend to zero as $r \to \infty$.

$$\sin b^n \pi (x + h_1{}^r) = \sin \frac{\pi}{2} = 1, \text{ and } \sin b^n \pi (x + h_2{}^r) = 0$$

and if $n > r$,

$$\sin b^n \pi (x + h_1{}^r) = +1 \text{ and } \sin b^n \pi (x + h_2{}^r) = 0.$$

Proceeding in a similar manner, *viz.*, breaking up $f(x)$ into two portions $\phi_1(x)$ and $\phi_2(x)$, we obtain

$$\left| \phi_1(x) \right| < \frac{a^r b^r \pi}{ab-1}$$

and $$\phi_2(x) = -\frac{2a^r b^r}{1-a}.$$

Hence $$\underset{r \to \infty}{\text{Lt}} \frac{f(x + h_1{}^r) - f(x + h_2{}^r)}{h_1{}^r - h_2{}^r} = -\infty$$

if $$\frac{2a^r b^r}{1-a} < \frac{a^r b^r \pi}{ab-1}$$

or if $$ab > 1 + \frac{\pi}{2}(1-a).$$

If we now choose $h_1{}^r$ and $h_2{}^r$ in the following manner, *viz.*, $b^r x = k_r - \varepsilon_r$ where $k_r = 2p - \frac{1}{2}$ where p is an integer and $0 \leqslant \varepsilon_r \leqslant 2$, we get the same limit to be $+\infty$. Hence under this condition the right-hand derivate does not exist.

We can also shew that the left-hand derivate is non-existent by putting $b^r x = k_r + \varepsilon_r$, $h_1{}^r = \frac{-\varepsilon_r}{b^r}$, $h_2{}^r = \frac{-\frac{1}{2} - \varepsilon_r}{b^r}$ and $k_r = 2p + \frac{1}{2}$ in one case and $2p - \frac{1}{2}$ in the other case.

(ii) Let $b = (4m - 1)$.

Let $b^r x = k_r - \varepsilon_r$ where $k_r = 2p + \frac{1}{2}$ and let

$$h_1{}^r = \frac{\varepsilon_r}{b^r} \text{ and } h_2{}^r = \frac{\varepsilon_r + \frac{1}{2}}{b^r}.$$

In this case $\sin b^n \pi (x + h_2^r) = 0$

if $n \geqslant r$ and $\sin b^{r+2p} \pi (x + h_2^r) = -1$, and $\sin b^{r+2p+1} \pi (x + h_2^r) = 1$ where p is an integer. Hence $\phi_2 (x)$ is, in this case, equal to

$$-2a^r b^r (1 - a + a^2 - ...) = \frac{-2a^r b^r}{1 + a}.$$

Hence the limit is $-\infty$ if $\frac{2a^r b^r}{1 + a} > \frac{a^r b^r \pi}{ab - 1}$ or if $ab > 1 + \frac{\pi}{2} (1 + a)$.

By choosing h_1^r and h_2^r in the following manner, *viz.*,

$b^r x = k_r - \varepsilon_r$ where $k_r = 2n + \frac{1}{2}$ and $0 \leqslant \varepsilon_r \leqslant 2$ and $h_1^r = \frac{\varepsilon_r}{b^r}$ and $h_2^r = \frac{\varepsilon_r + \frac{1}{2}}{b^r}$, we can shew that the limit is $+\infty$. Hence this shews that the right-hand derivate does not exist under the same condition. By putting

$$b^r x = k_r + \varepsilon_r, \quad h_1^r = \frac{-\varepsilon^r}{b^r}, \quad h_2^r = \frac{-\varepsilon_r - \frac{1}{2}}{b^r},$$

we can shew similarly that the left-hand derivate also does not exist under the same condition.

(iii) Let b be an even integer. Then put $b^r x = k_n - \varepsilon_r$ where $k_r = 2n + \frac{1}{2}$ and take

$$h_1^r = \frac{\varepsilon_r}{b^r} \text{ and } h_2^r = \frac{\varepsilon_r + \frac{1}{2}}{b^r}.$$

Then $\sin b^n \pi (x + h_1^r) = 1$ and $\sin b^n \pi (x + h_2^r) = 0$ if $n \geqslant r$ and $\sin b^n \pi (x + h_1^r) = 0$ if $n > r$. Hence in this case, $\phi_2(x) = \frac{a^r b^r}{2}$.

Therefore the limit will tend to $-\infty$ if $\frac{1}{2} > \frac{\pi}{ab - 1}$ or if $ab > 1 + \frac{\pi}{2}$. Proceeding in a similar manner we can shew that the right-hand and left-hand derivates do not exist.

ON CERTAIN DIRICHLET'S SERIES.

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1. The object of this paper is to study the following Dirichlet's series, and to obtain expressions of functions represented by them :—

$$\sum_{0}^{\infty} e^{Ai(n+a)^{\beta}} (n+a)^{-s}, \quad 0 < a < 1^{*},\ 0 < \beta \leqslant 1, \ldots \quad (1.1)$$

$$\left.\begin{array}{l} \displaystyle\sum_{0}^{\infty} e^{Ai(n+a)^{\beta} - s(n+a)^{\alpha}}, \\ \qquad 0 < \alpha < \beta, \end{array}\right\} \quad A \neq 0, \text{ when } \beta < 1, \text{ and } 0 < A < 2\pi, \text{ when } \beta = 1; \ldots \quad (1.2)$$

$$\sum_{0}^{\infty} \psi(n+a)\, e^{-s(n+a)^{\alpha}} \quad 0 < a < 1,\ 0 < \alpha < 1, \quad \ldots \quad (1.3)$$

$\psi(z)$ being an integral function of z, of order inferior to α.

With the exception of a particular case of series (1.1) which was considered by Lerch†, I do not think that these series were ever considered before.

2. Let us begin with the series (1.2). It is convergent if and only if $\sigma > 0$, and then absolutely convergent. In the half-plane $\sigma \geqslant \delta > 0$, ($\delta$ being an arbitrarily small positive number), the series represents an analytic function of s, say, $H(s)$. On the line $\sigma = 0$, the series is summable by Cesaro's means. It is not summable (C) for any value of s whose real part is negative, and the Riesz's typical means entirely fail to effect the analytic continuation of the function $H(s)$, which, nevertheless, is an integral function of s, as will be seen presently.

Suppose, for a moment, that s is real and positive. Let $A > 0$‡,

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* I have considered series (1.1) and (2.2) with $a = 1$ in my paper 'On the sum-functions of two Dirichlet's series, *The Tohoku Mathematical Journal*, Vol. 29, Nos. 3, 4.

† *Acta Mathematica*, Vol. XI. See also Whittaker and Watson, *Modern Analysis*, third edition, p. 280.

‡ The modification required for the case $A < 0$ will be obvious.

and suppose $(z+a)^{\alpha}$ and $(z+a)^{\beta}$ have their principal values. Then it is easy to prove* that

$$H(s) = \sum_{0}^{\infty} e^{Ai(n+a)^{\beta} - s(n+a)^{\alpha}} = \int_{-a}^{-a+i\infty} \frac{e^{Ai(z+a)^{\beta} - s(z+a)^{\alpha}}}{1 - e^{2\pi iz}}\, dz$$

$$+ \int_{-a}^{-a-i\infty} \frac{e^{Ai(z+a)^{\beta} - s(z+a)^{\alpha}}}{e^{2\pi iz} - 1}\, dz\,.$$

$$= \int_{-a}^{-a+i\infty} e^{Ai(z+a)^{\beta} - s(z+a)^{\alpha}}\, dz + i\int_{0}^{\infty} \frac{e^{Ai(it)^{\beta} - s(it)^{\alpha}}}{e^{2\pi ai + 2\pi t} - 1}\, dt\,.$$

$$- i\int_{0}^{\infty} \frac{e^{Ai(-it)^{\beta} - s(-it)^{\alpha}}}{e^{-2\pi ai + 2\pi t} - 1}\, dt$$

$$\equiv I_1 + I_2 + I_3, \text{ say.} \qquad \ldots \quad (2.1)$$

Let us first consider I_1. If $\beta = 1$, then

$$I_1 = i\int_{0}^{\infty} e^{-At - s(it)^{\alpha}}\, dt. \qquad \ldots \quad (2.2)$$

If $\beta < 1$, then

$$I_1 = e^{i\psi_1} \int_{0}^{\infty} e^{Ai(\rho e^{i\psi_1})^{\beta} - s(\rho e^{i\psi_1})^{\alpha}}\, d\rho, \qquad \ldots \quad (2.3)$$

where $\psi_1 = \dfrac{\pi}{2\beta}$; since

$$\left| \int_{\pi/2}^{\psi_1} \rho\, e^{Ai(\rho e^{i\psi})^{\beta} - s(\rho e^{i\psi})^{\alpha}} . e^{\psi i} i\, d\psi \right|$$

$$\leqslant \rho \int_{\pi/2}^{\psi_1} e^{-A\rho^{\beta} \sin \beta\psi - s\rho^{\alpha} \cos \alpha\psi}\, d\psi$$

$$\leqslant \rho \int_{\pi/2}^{\psi_1} e^{-A\rho^{\beta} \sin \beta\psi}\, d\psi$$

$$\leqslant \rho \int_{\pi/2}^{\psi_1} e^{-A\rho^{\beta} \beta\psi\, 2/\pi}\, d\psi$$

$$= \rho^{(1-\beta)} \left\{ \frac{e^{-A\rho^{\beta} . \beta} - e^{-A\rho^{\beta}}}{A\beta\, 2/\pi} \right\} \to 0$$

as $\rho \to \infty$.

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* I suppress the details of the proof, which rests merely on an application of Cauchy's theorem entirely similar to many others already made by me. See for example, my paper, 'On the analytic continuation of functions represented by a class of Dirichlet's series,' *The Journal of the Indian Mathematical Society*, Vol. XVII, p. 137 and p. 142.

It follows, therefore, that

$$I_1 = e^{i\pi/2\beta} \int_0^{\infty} e^{-A\rho^{\beta} - s\left(\rho e^{i\pi/2\beta}\right)^{\alpha}} d\rho$$

$$= \frac{e^{i\pi/2\beta}}{\beta} \int_0^{\infty} e^{-At - s\left(te^{i\pi/2}\right)^{\alpha/\beta}} . t^{(-1+1/\beta)} dt, \qquad \ldots (2.4)$$

whether $\beta < 1$, or $\beta = 1$.

Now expanding $e^{-s\left(te^{i\pi/2}\right)^{\alpha/\beta}}$ and integrating term by term, which is justified by the fact that

$$\int_0^{\infty} e^{-At} \sum_0^{\infty} \left| \frac{(-s)^n}{n!} t^{(n\alpha+1/\beta-1)} e^{i\pi(n\alpha+1)/2\beta} \right| dt$$

is convergent, we have

$$I_1 = \frac{1}{\beta} \sum_0^{\infty} \frac{(-s)^n}{n!} \Gamma\left(\frac{n\alpha+1}{\beta}\right) \left(Ae^{-i\pi/2}\right)^{-(n\alpha+1)/\beta}, \ldots (2.5)$$

which is easily seen to be an integral function of s.

Next, consider $I_2 + I_3$.

$$I_2 + I_3 = i \int_0^{\infty} \frac{e^{Ai(it)^{\beta} - s(it)^{\alpha}}}{e^{2\pi ai + 2\pi t}} dt - i \int_0^{\infty} \frac{e^{Ai(-it)^{\beta} - s(-it)^{\alpha}}}{e^{-2\pi ai + 2\pi t} - 1} dt,$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(-s)^n}{n!} \frac{(Ai)^{\nu}}{\nu!} \left\{ e^{i\pi(n\alpha+\nu\beta+1)/2} \int_0^{\infty} \frac{t^{(n\alpha+\nu\beta)} dt}{e^{2\pi ai + 2\pi t} - 1} \right.$$

$$\left. + e^{-i\pi(n\alpha+\nu\beta+1)/2} \int_0^{\infty} \frac{t^{(n\alpha+\nu\beta)} dt}{e^{-2\pi ai + 2\pi t} - 1} \right\}, \quad \ldots (2.6)$$

since

$$\int_0^{\infty} \left| \frac{1}{e^{\pm 2\pi ai + 2\pi t} - 1} \right| \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \left| \frac{(-s)^n}{n!} \right| \left| \frac{(Ai)^{\nu}}{\nu!} \right| \left| t \right|^{(n\alpha+\nu\beta)} dt$$

converges like the integral

$$\int_0^{\infty} e^{-2\pi t + |s| t^{\alpha} + |A| t^{\beta}} dt.$$

Hence

$$I_2 + I_3 = i \int_0^\infty \left\{ \frac{1}{e^{2\pi a i + 2\pi t} - 1} - \frac{1}{e^{-2\pi a i + 2\pi t} - 1} \right\} dt$$

$$+ \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \left\{ e^{\pi i (n\alpha + \nu\beta + 1)/2} \int_0^\infty t^{(n\alpha + \nu\beta)} \times \right.$$

$$\left(\sum_{\mu=1}^{\infty} e^{(-2\pi a i - 2\pi t)\mu} \right) dt$$

$$\left. + e^{-\pi i (n\alpha + \nu\beta + 1)/2} \int_0^\infty t^{(n\alpha + \nu\beta)} \left(\sum_{\mu=1}^{\infty} e^{(2\pi a i - 2\pi t)\mu} \right) dt \right\}, \qquad \ldots \quad (2.7)$$

$\Sigma' \Sigma'$ denoting that the term for which both n and ν are simultaneously zero is excluded from the double series.

Now $$i \int_0^\infty \left\{ \frac{1}{e^{2\pi a i + 2\pi t} - 1} - \frac{1}{e^{-2\pi a i + 2\pi t} - 1} \right\} dt$$

$$= 2 \int_0^\infty \frac{\sin 2\pi a}{e^{2\pi t} - 2 \cos 2\pi a + e^{-2\pi t}} dt = \zeta(0, a)^*,$$

where $\zeta(s, a)$ is the generalized zeta-function.

Also

$$\sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \sum_{\mu=1}^{\infty} \left| \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} e^{\pm i (n\alpha + \nu\beta + 1)\pi/2 \mp 2\pi\mu a i} \right| \times$$

$$\int_0^\infty e^{-2\pi\mu t} t^{(n\alpha + \nu\beta)} dt,$$

$$= \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \sum_{\mu=1}^{\infty} \left| \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \right| \frac{\Gamma(1 + n\alpha + \nu\beta)}{(2\pi\mu)^{(1 + n\alpha + \nu\beta)}},$$

* This follows from a formula for $\zeta(s, a)$, valid for $\sigma < 1$, given by Lindelöf in his "*Calcul des résidus*" on p. 107.

is convergent, hence the inversion of order of summation and integration in (2.7) is justified,* and we have,

$$I_2 + I_3 = \zeta(0, a) + \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \sum_{\mu=1}^{\infty} \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \times$$

$$\cdot \left\{ e^{i(n\alpha+\nu\beta+1)\pi/2 - 2\pi\mu a i} + e^{-i(n\alpha+\nu\beta+1)\pi/2+2\pi\mu a i} \right\} \times$$

$$\cdot \int_0^\infty e^{-2\pi\mu t}\, t^{(\alpha n+\nu\beta)}\, dt$$

$$= \zeta(0, a) + \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \frac{\Gamma(1+n\alpha+\nu\beta)}{(2\pi)^{(1+n\alpha+\nu\beta)}} \times$$

$$\left\{ e^{i(n\alpha+\nu\beta+1)\pi/2} \sum_{\mu=1}^{\infty} \frac{(\cos 2\pi\mu a - i\sin 2\pi\mu a)}{\mu^{(1+n\alpha+\nu\beta)}} \right.$$

$$\left. + e^{-i(n\alpha+\nu\beta+1)\pi/2} \sum_{\mu=1}^{\infty} \frac{(\cos 2\pi\mu a + i\sin 2\pi\mu a)}{\mu^{(1+n\alpha+\nu\beta)}} \right\}$$

$$= \zeta(0, a) + \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \frac{2\Gamma(n\alpha+\nu\beta+1)}{(2\pi)^{(1+n\alpha+\nu\beta)}} \times$$

$$\left\{ \sin(-n\alpha-\nu\beta)/2 \sum_{\mu=1}^{\infty} \frac{\cos 2\pi\mu a}{\mu^{(1+n\alpha+\nu\beta)}} \right.$$

$$\left. + \cos(-n\alpha-\nu\beta)\pi/2 \sum_{\mu=1}^{\infty} \frac{\sin 2\pi\mu a}{\mu^{(1+n\alpha+\nu\beta)}} \right\}$$

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* Bromwich, '*Infinite Series*', second edition, pp. 500-501.

$$= \zeta(0, a) + \sum_{n=0}^{\infty}{}' \sum_{\nu=0}^{\infty}{}' \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \zeta(-n\alpha - \nu\beta, a)^*$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(-s)^n}{n!} \frac{(Ai)^\nu}{\nu!} \zeta(-n\alpha - \nu\beta, a), \qquad \ldots \quad (2.8)$$

which can be easily shown to represent an integral function of s.

It follows, therefore, that

$$H(s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \left\{ \frac{1}{\beta} \Gamma\left(\frac{1+n\alpha}{\beta}\right) \left(Ae^{-i\pi/2} \right)^{-\frac{n\alpha+1}{\beta}} \right.$$

$$\left. + \sum_{\nu=0}^{\infty} \frac{(Ai)^\nu}{\nu!} \zeta(-n\alpha - \nu\beta, a) \right\} \qquad \ldots \quad (2.9)$$

This equation has been obtained on the assumption that s is real and positive. But as the right side represents an integral function of s, the equation persists for all values of s. H (s) is, therefore, an integral function of s, and is given by the formula (2.9). If we put $a = 1$ in (2.9), then we have the result obtained in my paper already referred to.

For the sake of completeness, I may give the function represented by the series (1.2), when $0 < \beta < \alpha < 1$, $A \neq 0$.

This function is found to be

$$\sum_{n=0}^{\infty} \frac{(Ai)^n}{n!} \left\{ \frac{1}{\alpha} \frac{\Gamma\left(\dfrac{1+n\beta}{\alpha}\right)}{s^{(1+n\beta)/\alpha}} + \sum_{\nu=0}^{\infty} \frac{(-s)^\nu}{\nu!} \zeta(-\alpha\nu - \beta n, a) \right\},$$

$$\ldots \quad (2.10)$$

where the double series represents an integral function of s, and the single series $= (s^{-1/\alpha}) \times$ (an integral function of $(s^{-\beta/\alpha})$).

Lastly, suppose $\phi(A, s) = \sum_{1}^{\infty} e^{Ain^\beta - sn^\alpha}$,

and

$$\phi(A, s, a) = \sum_{0}^{\infty} e^{Ai(n+a)^\beta - s(n+a)^\alpha},$$

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* By the formula of Hurwitz. See Whittaker and Watson, *loc. cit.*, p. 269.

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NOTES AND QUESTIONS.

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AND

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Notes and Questions.

A General Form of the Remainder in Taylor's Theorem.

If $f(x)$, $g(x)$, $h(x)$ be continuous functions, each of which has derivatives of the first and second orders, and if $0 < k < l$, then we have

$$\begin{vmatrix} f(a), & g(a), & h(a) \\ f(a+k), & g(a+k), & h(a+k) \\ f(a+l), & g(a+l), & h(a+l) \end{vmatrix}$$

$$-\frac{1}{1.2}\begin{vmatrix} f(a), & g(a), & h(a) \\ f'(\xi), & g'(\xi), & h'(\xi) \\ f''(\eta), & g''(\eta), & h''(\eta) \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & a+k & (a+k)^2 \\ 1 & a+l & (a+l)^2 \end{vmatrix} = 0 \quad \ldots \quad (1)$$

where $\qquad a < \xi < a + k\,;\ a < \eta < a + l.$

[See Goursat-Hedrick : *Math. Anal.* Vol. I, pp. **10-11**.]

Suppose $F(x)$, $G(x)$ and $H(x)$ to be continuous functions whose derivatives up to the $(p+1)$th, $(q+1)$th and $(r+1)$th orders respectively exist and such that $F^{p+1}(x)$, $G^{q+1}(x)$, $H^{r+1}(x)$ do not vanish, but that their higher derived functions do so. If R_p is the remainder after $p+1$ terms in Taylor's expansion of $F(a+k)$ then

$$R_p = F(a+k) - F(a) - kF'(a) - \ldots - \frac{k^p}{p!}F^p(a). \quad \ldots \quad (2)$$

Let $f(x)$, $g(x)$, $h(x)$ be functions defined as follows :

$$\left.\begin{aligned} f(x) &= F(a+k) - F(x) - (a+k-x)F'(x) - \ldots \\ &\qquad - \frac{(a+k-x)^p}{p!}F^p(x), \\ g(x) &= G(a+k) - G(x) - (a+k-x)G'(x) - \ldots \\ &\qquad - \frac{(a+k-x)^q}{q!}G_q(x), \\ h(x) &= H(a+k) - H(x) - (a+k-x)H'(x) - \ldots \\ &\qquad - \frac{(a+k-x)^r}{r!}H^r(x). \end{aligned}\right\} \quad . \quad (3)$$

Then $f(x)$, $g(x)$ and $h(x)$ satisfy the conditions required above.

Moreover

$$f(a) = R_p \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

$$f(a+k) = 0\,; \quad g(a+k) = 0\,; \quad h(a+k) = 0 \qquad \ldots \qquad (5)$$

$$\left.\begin{aligned} f'(x) &= -\frac{(a+k-x)^p}{p\,!}\,F^{p+1}(x) \\ g'(x) &= -\frac{(a+k-x)^q}{q\,!}\,G^{q+1}(x) \\ h'(x) &= -\frac{(a+k-x)^r}{r\,!}\,H^{r+1}(x) \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (6)$$

and

$$\left.\begin{aligned} f''(x) &= \frac{(a+k-x)^{p-1}}{(p-1)\,!}\,F^{p+1}(x) \\ g''(x) &= \frac{(a+k-x)^{q-1}}{(q-1)\,!}\,G^{q+1}(x) \\ h''(x) &= \frac{(a+k-x)^{r-1}}{(r-1)\,!}\,H^{r+1}(x) \end{aligned}\right\} \qquad \ldots \qquad (7)$$

The first determinant in (1) by virtue of (5) vanishes, and the second determinant of the product is not equal to zero. Substituting from (4), (5) and (6) and putting $\xi = a + \theta_1 k\,;\ \eta = a + \theta_2 l$, where θ_1 and θ_2 are positive proper fractions, we get

$$\begin{vmatrix} R_p, & g(a), & h(a) \\ \dfrac{k^p(1-\theta_1)^p}{p\,!}\,F^{p+1}(\xi), & \dfrac{k^q(1-\theta_1)^q}{q\,!}\,G^{q+1}(\xi), & \dfrac{k^r(1-\theta_1)^r}{r\,!}\,H^{r+1}(\xi) \\ \dfrac{(k-\theta_2 l)^{p-1}}{(p-1)\,!}\,F^{p+1}(\eta) & \dfrac{(k-\theta_2 l)^{q-1}}{(q-1)\,!}\,G^{q+1}(\eta), & \dfrac{(k-\theta_2 l)^{r-1}}{(r-1)\,!}\,H^{r+1}(\eta) \end{vmatrix} = 0$$

where $g(a)$ and $h(a)$ have values given by (3).

From this relation R_p is found.

Since expressions analogous to (1) exist for n functions, the proof following the same lines, in the same way we may obtain the relation

$$\begin{vmatrix} R_p, & g(a) & h(a) & j(a) \\ \frac{(k-\theta_1 k)^p}{p!} F^{p+1}(\xi), & \frac{(k-\theta_1 k)^q}{q!} G^{q+1}(\xi) & \dots & \dots \\ \frac{(k-\theta_2 l)^{p-1}}{(p-1)!} F^{p+1}(\eta), & \frac{(k-\theta_2 l)^{q-1}}{(q-1)!} G^{q+1}(\eta) & \dots & \dots \\ \frac{(k-\theta_3 n)^{p-2}}{(p-2)!} F^{p+1}(\zeta), & \frac{(k-\theta_3 n)^{q-2}}{(q-2)!} G^{q+1}(\zeta) & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

with analogous restrictions on the functions.

The determinant is restricted to order $n + 2$ where n is the least of the quantities p, q, r, etc. Hence we may not have more than $n+2$ functions.

If only two functions $F(x)$ and $G(x)$ are taken the formulae reduce to

$$R_p = \frac{q!\,(k-\theta_1 k)^{p-q}\,F^{p+1}(a+\theta_1 k)}{p!\,G^{p+1}(a+\theta_1 k)}\cdot g(a),$$

where

$$g(a) = G(a+k) - G(a) - kG'(a) - \frac{k^2}{2!}G''(a) - \dots - \frac{k^q}{q!}G^q(a).$$

If n is written for p, and p for q, we have the formula given by G. S. Mahajani in "*The Messenger of Mathematics*," Vol. 52, No. 5, Sep. 1922. From it he derives Lagrange's, Cauchy's, Schlomilch's and other forms of the remainder.

J. B. FREEMAN.

The Gauss-Mainardi-Codazzi Equations.

Let r be the vector to any point (x, y, z) of a surface whose curvilinear co-ordinates are x_1 and x_2. (It is convenient to use x_1 and x_2 instead of p, q or u, v) Let the suffixes 1 and 2 be used to denote differentiation with respect to x_1 and x_2 respectively.

By the rule for the differentiation of a scalar product, we have

$$\frac{\partial}{\partial x_\beta}(\omega . r_{\alpha\beta}) = \omega_\beta . r_{\alpha\beta} + \omega . r_{\alpha\beta\beta},$$

where $\alpha, \beta = 1, 2$, and ω is any vector.

or, $$-\omega . r_{\alpha\beta\beta} = -\frac{\partial}{\partial x_\beta}(\omega . r_{\alpha\beta}) + \omega_\beta . r_{\alpha\beta}.$$

Similarly

$$-\omega . r_{\alpha\beta\beta} = -\frac{\partial}{\partial x_\alpha}(\omega . r_{\beta\beta}) + \omega_\alpha . r_{\beta\beta}$$

Therefore

$$-\frac{\partial}{\partial x_\beta}(\omega . r_{\alpha\beta}) + \omega_\beta . r_{\alpha\beta} = -\frac{\partial}{\partial x_\alpha}(\omega . r_{\beta\beta}) + \omega_\alpha . r_{\beta\beta} \quad \ldots \quad \text{(A)}$$

The two Mainardi-Codazzi equations are obtained by putting in (A), ω equal to the unit vector n along the normal and $\alpha, \beta = 1, 2$ or $2, 1$. The Gauss characteristic equation is obtained *twice* by putting $\omega = r_\alpha$, $(\alpha, \beta) = (1, 2)$, or $(2, 1)$. If we put $\omega = r_\beta$, we get identities.* $\alpha = \beta$ gives no results.

Thus

$$-\frac{\partial}{\partial x_2}(n . r_{12}) + (n_2 . r_{12}) = -\frac{\partial}{\partial x_1}(n . r_{22}) + (n_1 . r_{22}) \quad \ldots \quad (1)$$

$$-\frac{\partial}{\partial x_1}(n . r_{21}) + (n_1 . r_{21}) = -\frac{\partial}{\partial x_2}(n . r_{11}) + (n_2 . r_{11}) \quad \ldots \quad (2)$$

$$-\frac{\partial}{\partial x_2}(r_1 . r_{12}) + r^2_{12} = -\frac{\partial}{\partial x_1}(r_1 . r_{22}) + (r_{11} . r_{22}) \quad \ldots \quad (3)$$

---

* Using the notation of Forsyth's *Differential Geometry*, these identities reduce to $m'_1 = m_2$ and $n''_1 = n'_2$.

or $$-\frac{\partial}{\partial x_1}(r_2 . r_{21}) + r_{21}^2 = -\frac{\partial}{\partial x_2}(r_2 . r_{11}) + (r_{22} . r_{11}) \quad \ldots \quad (3')$$

are the Mainardi-Codazzi and the Gauss equations respectively.

We shall not work out here in detail their conversion to the familiar Cartesian forms. They may be obtained by using on the two sides of the equations, the Gaussian formulae

$$r_{11} = L . n + \Gamma . r_1 + \Delta . r_2$$

$$r_{12} = M . n + \Gamma' . r_1 + \Delta' . r_2$$

$$r_{22} = N . n + \Gamma'' . r_1 + \Delta'' . r_2$$

and

$$(n . r_{11}) = -(n_1 . r_1) = L$$

$$(n . r_{12}) = -(n_1 . r_2) = -(n_2 . r_1) = M$$

$$(n . r_{22}) = -(n_2 . r_2) = N$$

(*Vide* Forsyth's ***Differential Geometry,*** §§ 29 & 34).

Substituting for r_{11}, r_{12}, r_{21} on both sides of the equations, we easily obtain

$$M_2 + \Gamma' M + \Delta' N = N_1 + \Gamma'' L + \Delta'' M$$

$$L_2 + \Gamma M + \Delta N = M_1 + \Gamma' L + \Delta' M$$

$$LN - M^2 = -\tfrac{1}{2}(E_{22} - 2F_{12} + G_{11}) + (E, F, G)(\Gamma', \Delta')^2$$
$$- (E, F, G)(\Gamma, \Delta)(\Gamma'', \Delta'').$$

H. BALAKRAM.

[The above note is based upon a letter to me from the late Mr. Balakram, written shortly before his sudden death. The proof of the fundamental identities connecting E, F, G, L, M, N is, as Balakram observes, not *essentially* different from the usual proofs, but by using the vector notation he obtains a noteworthy amount of compactness. It is to be observed that, but for the work of conversion from the vectorial form to the Cartesian form, "the equations are so easily obtained that they hardly deserve a special name." Balakram remarks that

all "the three equations are the consequences of the fact (or assumption) that the operators $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ are commutative, or that $dx_1 + dx_2 = dx_2 + dx_1$."*

Balakram was probably not well acquainted with books on Differential Geometry using the vector notation. Since receiving his letter, I have compared his method and form of statement with that given in C. E. Weatherburn's "Differential Geometry of Three Dimensions" and W. Blaschke's "Vorlesungen uber Differential Geometrie I" and I feel some justification for sending this note to the press. ***It is to be noted that the single identity (A) includes all the three equations of Gauss, Mainardi and Codazzi.*** Blaschke gives an equally comprehensive formula, but rather different in form and principle.

I have preferred to make a slight change of notation using the more modern notation $a \cdot b$ to denote the scalar product of the vectors a and b, which Balakram writes in the quaternionic form S ab.

C. N. SRINIVASIENGAR.

The Equation of Geodesics on a Developable Surface.

The general equation of Geodesics on a developable surface may be expressed in terms of the curvilinear co-ordinates r and ψ in the form

$$r \sin(\psi - c_1) + \int \rho \sin(\psi - c_1)\, d\psi = c_2,$$

where c_1 and c_2 are arbitrary constants,

r is the distance measured along a generator from its point of contact with the edge of regression,

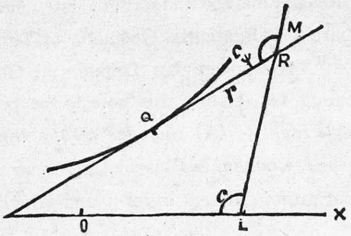
and ρ is the radius of curvature of the edge at that point

$$\left(\rho = \frac{ds}{d\psi}\right).$$

---

* Is the converse, *viz.*, Bonnet's fundamental theorem also the direct consequence of this or an equally simple postulate?—C. N. S.

The simplest proof of this result depends on the property that every developable surface can be opened out into a plane whereby the Geodesics are deformed into straight lines. The edge of regression is converted into a plane curve whose tangents correspond to the generators and whose curvature is equal to the circular curvature of the edge at the corresponding point.



Let C denote this plane curve and Q the point corresponding to a point P on the edge. Let any geodesic on the original surface correspond to the straight line LRM. Then r is equal to QR. Let $\angle QRM = \psi$. Choosing any set of rectangular axes in the plane, let $Q \equiv (x, y)$. Then the co-ordinates of R are

$$\xi = x + r\cos(\psi - c)$$

$$\eta = y + r\sin(\psi - c), \text{ where } c = O\hat{L}M.$$

But $$\frac{dx}{ds} = \cos(\psi - c), \quad \frac{dy}{ds} = \sin(\psi - c).$$

Therefore,

$$\xi = r\cos(\psi - c) + \int \rho\cos(\psi - c)\, d\psi$$

$$\eta = r\sin(\psi - c) + \int \rho\sin(\psi - c)\, d\psi.$$

Since (ξ, η) satisfies an equation of the form $A\xi + B\eta + C = 0$, we obtain the relation

$$r\sin(\psi - c_1) + \int \rho\sin(\psi - c_1)\, d\psi = c_2.$$

This relation must therefore be satisfied by the geodesic on the original developable.

The geometrical interpretation for ψ, *viz.*, that it is the angle between the generator and the tangent to the geodesic, may be noted.

It is possible to obtain these results directly without the use of the deformation process, but the work is slightly longer and less interesting. The problem discussed here serves to show how geometrical methods may be often used in analysis with great advantage, for if we were to write down the general differential equations of geodesics in curvilinear co-ordinates for the present case, the equation is very complicated, and no simple way of integration suggests itself.

NOTE.—The case where the developable is a cone requires no separate discussion. The edge of regression consists of a point only. When the cone is opened out into a plane, this point may be taken as the origin and the rest of the above discussion may be repeated. The geodesics are given by $r \sin (\psi - c_1) = c_2$.

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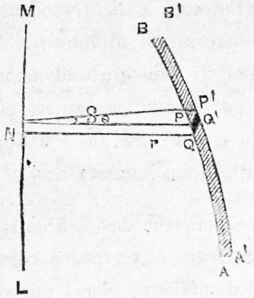
K. S. K. IYENGAR.

A Note on Centre of Gravity.

In the 'Note on Centre of Gravity' published in this *Journal* for October 1930, it has been proved that the distance of the centre of gravity of a surface or a solid of revolution from the axis of rotation is of the form $f(2\theta)$ and is equal to $a \sin \theta/\theta$ where 2θ is the angle of rotation and a is the limit of $f(x)$ as x tends to zero. Also it was stated there that to find the value of the constant a, we have to know the position of the centre of gravity for some value of θ say $\theta = \pi/2$. But in this note I shall show that a is equal to $k^2/\bar{z}$ where Mk^2 is the moment of inertia about the axis of rotation of the arc or the area which generates the surface or the solid and $\bar{z}$ is the distance of the centre of gravity of the same arc or the area from the axis of rotation.

It is obvious that the value of a only depends on the elements of the arc or the area according as we are considering either the surface or the solid of revolution. First I will consider the surface generated by an arc AB. Since we require the position of the centre of gravity in the limiting position, *i e*, when the angle of rotation is very small, let the arc AB be rotated through an angle $\delta\theta$, and the surface generated



be AA′B′B, the axis of rotation being LM. Suppose P and Q are any two consecutive points on the arc AB and the surface generated by PQ is PP′Q′Q. Area of this small element may be taken to be equal to $\delta s . r\delta\theta$ where δs is the length of the arc PQ, and r is the distance of P from LM. So far as the centre of gravity of this element area is concerned, we may replace the mass contained in it by a particle of equivalent mass at P. This process is justified because we want the position of the centre of gravity only when $\delta\theta$ is infinitesimal. Now the mass at P is proportional to the area PP′Q′Q, *i e*, to the expression $r . \delta s . \delta\theta$. Since $\delta\theta$ is the same for all positions of P on the arc AB we may take the mass at P to be proportional to $r . \delta s$. Therefore finding the value of a, that is, the distance of the centre of gravity of the surface generated in the limiting position as $\delta\theta \rightarrow 0$ is reduced to one of finding out the centre of gravity of the arc AB, on the assumption that the density at any point on the arc varies as its distance from the axis of rotation.

$$\therefore \quad a = \frac{\int r^2 \, ds}{\int r \, ds}$$

the limits of the integrals being so chosen as to include the whole length of the arc AB. But we know that $\int r^2\, ds = \sigma k^2$, σ being the length of the arc AB and

$$\int r\, ds = \sigma \bar{z}. \quad \therefore \quad a = \frac{k^2}{\bar{z}}.$$

The method of proof for the solid of revolution is exactly similar to the above; only we have to replace the arc by area and the surface by solid.

Now the equation giving the distance of the centre of gravity of the surface or the solid of revolution from the axis of rotation can be written in the form

$$f(2\theta) = \frac{k^2}{\bar{z}} \frac{\sin\theta}{\theta}.$$

The following simple examples illustrate the use of this equation.

i. An arc of a circle subtending an angle 2θ at the centre may be taken as one which has been generated by a point at a distance r from the centre, r being the radius and 2θ the angle of rotation. Here $k^2 = r^2$ and $\bar{z} = r$.

$$\therefore \quad f(2\theta) = \frac{k^2}{\bar{z}} \frac{\sin\theta}{\theta} = r \cdot \frac{\sin\theta}{\theta}.$$

ii. A sector of a circle of angle 2θ at the centre may be generated by rotating a straight line about one of its ends through an angle 2θ, the length of the line being equal to the radius of the circle. In this case $k^2 = r^2/3$ and $\bar{z} = r/2$.

$$\therefore \quad f(2\theta) = \frac{r^2}{3} \cdot \frac{2}{r} \cdot \frac{\sin\theta}{\theta} = \frac{2}{3} r \cdot \frac{\sin\theta}{\theta}.$$

iii. A semi-circular arc or area rotated through an angle 2θ. For the arc $k^2 = \frac{r^2}{2}$, $\bar{z} = \frac{2r}{\pi}$ and for the area $k^2 = \frac{r^2}{4}$ and $\bar{z} = \frac{4r}{3\pi}$.

$$\therefore \text{ for the surface of a lune of a sphere } f(2\theta) = \frac{r^2}{2} \cdot \frac{\pi}{2r} \cdot \frac{\sin\theta}{\theta}$$

$$= \frac{\pi r}{4} \cdot \frac{\sin\theta}{\theta}.$$

and for the solid $f(2\theta) = \frac{r^2}{4} \frac{3\pi}{4r} \cdot \frac{\sin\theta}{\theta}$

$$= \frac{3\pi r}{16} \cdot \frac{\sin\theta}{\theta}.$$

iv. Suppose a triangular area is rotated about a certain line in the plane of the triangle through an angle 2θ and α, β, γ are the distances of the mid.-points of the sides of the triangle from the axis of rotation.

Here $k^2 = \alpha^2 + \beta^2 + \gamma^2$ and $\bar{z} = \frac{\alpha + \beta + \gamma}{3}$.

$$\therefore \quad f(2\theta) = 3 \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha + \beta + \gamma} \cdot \frac{\sin\theta}{\theta}.$$

The above formula $f(2\theta) = \frac{k^2}{\bar{z}} \cdot \frac{\sin\theta}{\theta}$ is true even when the mass at any point on the arc or on the area is some function of the distance from the axis of rotation.

M. N. N.

Triangles Multiply Orthological.

1. Let ABC, DEF be two triangles in a plane and α, β, γ, be vectors perpendicular to the sides EF, FD, DE of the triangle DEF. The two triangles can be orthological in, at the most, six ways. Denoting by $A\alpha$ the line through A parallel to the vector α, one or more of the following relations may hold :—

$A\alpha$, $B\beta$, $C\gamma$	may concur in	P_1	... (1)
$A\beta$, $B\gamma$, $C\alpha$	,,	P_2	... (2)
$A\gamma$, $B\alpha$, $C\beta$	,,	P_3	... (3)
$A\alpha$, $B\gamma$, $C\beta$	,,	Q_1	... (4)
$A\gamma$, $B\beta$, $C\alpha$	,,	Q_2	... (5)
$A\beta$, $B\alpha$, $C\gamma$	,,	Q_3	... (6)

2. Suppose relations (1) and (2) subsist.

If the areal co-ordinates of P_1 with respect to Δ ABC be (λ, μ, ν), then the equations of the lines $A\alpha$, $B\beta$, $C\gamma$ are $\frac{y}{\mu} - \frac{z}{\nu} = 0$, etc. and the condition for the concurrence of the lines in (2) whose equations are of the type $y \frac{1}{\lambda} + z\left(\frac{1}{\nu} + \frac{1}{\lambda}\right) = 0$ becomes

$$\Sigma \mu\nu = 0.$$

This is also the condition that relation (3) may hold.

Hence P_1 lies on the minimum circumscribed conic

$$yz + zx + xy = 0 \qquad \dots \quad (7)$$

The co-ordinates of P_2 are (ν, λ, μ) and hence it also lies on the ellipse (7). Similarly P_3 also lies on the ellipse.

Hence when two triangles are doubly orthological in the cyclic order, they are also trebly orthological. Hence we can construct triangles trebly orthological in the cyclic order.

3. Let relations (1) and 4) subsist.

From a diagram, it becomes evident that BP_1CQ_1 is a parallelogram, that A, P_1, Q_1 are collinear, and that AP_1Q_1 is the median through A. Hence if P_1 is given, Q_1 can be found uniquely.

If the triangles are orthological in the form (5) also, then P_1 becomes the centroid ; Q_1, Q_2, Q_3 are the images of the vertices in the centroid and the triangles are orthological of the type (6) also.

Hence we can construct triangles orthological in four ways. The sides of the second triangle are perpendicular to the medians of the first and *vice versa.*

4. It is easy to see that two triangles cannot be orthological in more than four ways.

N. DURAIRAJAN.

On the Foci of a Conic.

The following is an alternative method for finding the foci,* the equations of the directrices and axes of the conic,

$$S(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots \quad (1)$$

Suppose (α, β) is a focus of (1) and transfer the origin to it. Then (1) reduces to

$$ax^2 + 2hxy + by^2 + x\frac{\partial S}{\partial \alpha} + y\frac{\partial S}{\partial \beta} + S(\alpha, \beta) = 0 \quad \dots \quad (2)$$

If the directrix of (2) with respect to (0, 0) is

$$lx + my + n = 0 \quad \dots \quad (3)$$

then (2) must be of the form

$$\lambda(x^2 + y^2) = (lx + my + n)^2 \quad \dots \quad (4)$$

Hence

$$ax^2 + 2hxy + by^2 + x\frac{\partial S}{\partial \alpha} + y\frac{\partial S}{\partial \beta} + S(\alpha, \beta) - \lambda(x^2 + y^2)$$

i.e., $$(a - \lambda)x^2 + 2hxy + (b - \lambda)y^2 + x\frac{\partial S}{\partial \alpha} + y\frac{\partial S}{\partial \beta} + S(\alpha, \beta) \quad (5)$$

must reduce to a perfect square for a suitable value of λ. For such values of λ

$$(a - \lambda)x^2 + 2hxy + (b - \lambda)y^2,$$

$$(a - \lambda)x^2 + x\frac{\partial S}{\partial \alpha} + S(\alpha, \beta),$$

$$(b - \lambda)y^2 + y\frac{\partial S}{\partial \beta} + S(\alpha, \beta)$$

must each be a perfect square. Hence

$$(a - \lambda)(b - \lambda) = h^2 \quad \dots \quad (6)$$

$$4(a - \lambda)S(\alpha, \beta) = \left(\frac{\partial S}{\partial \alpha}\right)^2 \quad \dots \quad (7)$$

$$4(b - \lambda)S(\alpha, \beta) = \left(\frac{\partial S}{\partial \beta}\right)^2 \quad \dots \quad (8)$$

---

* Cf. Askwith : *Analytical Geometry of the Conic Sections*, pp. 223-24.

Substituting from (6), (7), (8) in (5), expression (5) should reduce to

$$\frac{1}{(a-\lambda)}\left\{(a-\lambda)x + hy + \tfrac{1}{2}\frac{\partial S}{\partial \alpha}\right\}^2 \qquad \ldots \quad (9)$$

(If $h \neq 0$, then $a - \lambda \neq 0$).

Comparing the co-efficients of y in (5) and (9) we get

$$h\frac{\partial S}{\partial \alpha} = (a-\lambda)\frac{\partial S}{\partial \beta} \qquad \ldots \quad (10)$$

From (7), (8), (10) we immediately get equations which determine the foci, namely

$$\frac{\left(\frac{\partial S}{\partial \alpha}\right)^2 - \left(\frac{\partial S}{\partial \beta}\right)^2}{a-b} = \frac{\frac{\partial S}{\partial \alpha}\cdot\frac{\partial S}{\partial \beta}}{h} = 4\,S(\alpha, \beta) \qquad \ldots \quad (11)$$

Again $$lx + my + n = 0$$

is equivalent to

$$(a-\lambda)x + hy + \frac{1}{2}\frac{\partial S}{\partial \alpha} = 0 \qquad \ldots \quad (12)$$

Hence (12) is the directrix of the conic (2) with respect to (0, 0). Referring (12) to old axes we get the equation of the directrix of (1) with respect to the focus (α, β) in the form

$$ax + hy + g - \lambda(x - \alpha) = 0 \qquad \ldots \quad (13)$$

Also, (10) shows that the focus (α, β) lies on

$$h\frac{\partial S}{\partial x} = (a-\lambda)\frac{\partial S}{\partial y} \qquad \ldots \quad (14)$$

But (14) obviously passes through the centre of the conic (1). Hence (14) is the principal axis that passes through the focus (α, β).

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Dharwar.

B. B. Bagi.

Solutions.

Question 1469.

(S D. CHOWLA) :—Prove geometrically that

$$\frac{r^2}{2}\sin\frac{2\pi}{n}+\sum_{t=1}^{t=\infty}2^{t-1}x_t\left\{r-\sqrt{r^2-x^2_t/4}\right\}=\frac{\pi r^2}{n}$$

where
$$x_1 = 2r\sin\frac{\pi}{n}$$

$$x_{t+1}=\sqrt{2r^2-\sqrt{4r^4-r^2x^2_t}}$$

r being positive, and n a positive integer.

Solution by K. V. Krishna Sastri,

V. A. Mahalingam, S. V. Minakshisundaram and K. S. Subramanyam.

There is a slight mistake in the question. We have to read 2^{t-2} instead of 2^{t-1}.

Suppose AOB is a sector of a circle of radius r having its centre at O and having arc AB = nth of the circumference. Bisect the arc AB at B_1, the arc AB_1 at B_2, the arc AB_2 at B_3, and in general the arc AB_{t-1} at B_t. Let us denote the lengths of the chords

$$AB,\ AB_1,\ AB_2,\ \ldots\ AB_{t-1},\ \text{by}\ x_1,\ x_2,\ x_3,\ \ldots\ x_t,$$

respectively.

It is easy to see that $x_1 = 2r\sin\frac{\pi}{n}$.

For x_2 we have $x_2^2 = AB_1^2 = OA^2+OB_1^2-2OA.OB_1\cos A\hat{O}B_1$

$$= 2r^2-2r^2\,.\,\frac{ON_1}{r}$$

(where N_1 is the pt. at which OB_1 meets AB).

$$\therefore\qquad x_2^2 = 2r^2-2r\sqrt{r^2-x_1^2/4}.$$

$$\therefore\qquad x_2=\sqrt{2r^2-\sqrt{4r^4-x_1^2r^2}}$$

and in general we find that

$$x_{t+1} = \sqrt{2r^2 - \sqrt{4r^4 - r^2 x^2_t}}.$$

Now the area of the sector OAB of the circle is given by

$$S = \Delta\, OAB + \Delta\, AB_1B + 2\Delta\, AB_2B_1$$
$$+ 2^2\, \Delta\, AB_3B_2 + \ldots + 2^{t-1}\, \Delta\, AB_t\, B_{t-1} + \ldots \text{ ad. inf.}$$

Now $$\Delta\, OAB = \tfrac{1}{2}\, r_2 \sin \frac{2\pi}{n}$$

$$\Delta\, AB_1\, B = \tfrac{1}{2}\, AB\,.\,B_1N$$
$$= \tfrac{1}{2}\, x_1\, (r - ON)$$
$$= \tfrac{1}{2}\, x_1\, (r - \sqrt{r^2 - x_1^{\,2}/4}).$$

In general

$$\Delta\, AB_t\, B_{t-1} = \tfrac{1}{2}\, x_t\, (r - \sqrt{r^2 - x_t^{\,2}/4}).$$

Therefore

$$S = \frac{r^2}{2} \sin \frac{2\pi}{n} + \sum_{t=1}^{t=\infty} 2^{t-2}\, x_t\, (r - \sqrt{r^2 - x_t^{\,2}/4}).$$

But $S = \frac{\pi r^2}{n}$. Hence the result.

Question 1487.

(S. D. CHOWLA): Prove that

(i) $$1^2 + \frac{1^2}{x^2} + \frac{1^2\,(x+1)^2}{x^2\,(2x)^2} + \frac{1^2\,(x+1)^2\,(2x+1)^2}{x^2\,(2x)^2\,(3x)^2} + \ldots \quad \ldots$$

$$= \frac{1}{\pi} \sin \frac{\pi}{x}\, \frac{\Gamma\left(\frac{1}{x}\right)\, \Gamma\left(1 - \frac{2}{x}\right)}{\Gamma\left(1 - \frac{1}{x}\right)} \qquad (x > 2)$$

(ii) $$1^2 - \frac{1^2}{x^2} + \frac{1^2\,(x+1)^2}{x^2\,(2x)^2} - \frac{1^2\,(x+1)^2\,(2x+1)^2}{x^2\,(2x)^2\,(3x)^2} + \ldots \quad \ldots$$

$$= \frac{1}{2x} \sin \frac{\pi}{x}\, \frac{\Gamma\left(\frac{1}{2x}\right)\, \Gamma\left(1 - \frac{1}{x}\right)}{\Gamma\left(1 - \frac{1}{2x}\right)} \qquad (x > 1)$$

(iii) $$1+\frac{1^3}{2^3}x+\frac{1^3 . 3^3}{2^3 . 4^3}x^2+\frac{1^3 . 3^3 . 5^3}{2^3 . 4^3 . 6^3}x^3+\ldots$$

$$=\left(1+\frac{1^2}{4^2}x+\frac{1^2 . 5^2}{4^2 . 8^2}x^2+\frac{1^2 . 5^2 . 9^2}{4^2 . 8^2 . 12^2}x^3+\ldots\right)^2$$

$$(\,|\,x\,|\leqslant 1)$$

and deduce Ramanujan's result

$$1-\frac{1^3}{2^3}+\frac{1^3 . 3^3}{2^3\ 4^3}-\frac{1^3 . 3^3 . 5^3}{2^3 . 4^3 . 6^3}+\ldots=\left\{\frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4}) . \Gamma(\frac{7}{8})}\right\}^2$$

Solution by S Mahadevan and S. D. Chowla.

We know that

$$F(\alpha,\beta,\gamma,x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}\,dt \quad \ldots \quad (1)$$

where

$$F(\alpha,\beta,\gamma,x)=1+\frac{\alpha . \beta}{1 . \gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1 . 2 . \gamma\,(\gamma+1)}x^2+\ldots$$

$$(\alpha>0,\ \gamma-\alpha>0)$$

(*vide* Bromwich: *Infinite Series*, Chap. VIII).

Putting $x=1$ we get

$$F\left(\frac{1}{x},\frac{1}{x},1,1\right)=\frac{\Gamma(1)}{\Gamma\left(\frac{1}{x}\right)\Gamma\left(1-\frac{1}{x}\right)}\int_0^1 t^{\frac{1}{x}-1}(1-t)^{-\frac{2}{x}}\,dt.$$

But

$$\Gamma\left(\frac{1}{x}\right)\Gamma\left(1-\frac{1}{x}\right)=\frac{\pi}{\sin\pi/x}$$

and

$$\int_0^1 t^{\frac{1}{x}-1}(1-t)^{-\frac{2}{x}}\,dt=\frac{\Gamma\left(\frac{1}{x}\right)\Gamma\left(1-\frac{2}{x}\right)}{\Gamma\left(1-\frac{1}{x}\right)}.$$

$$\therefore\quad F\left(\frac{1}{x},\frac{1}{x},1,1\right)=\frac{1}{\pi}\sin\frac{\pi}{x}\,\frac{\Gamma\left(\frac{1}{x}\right)\Gamma\left(1-\frac{2}{x}\right)}{\Gamma\left(1-\frac{1}{x}\right)} \qquad (x>2)$$

which is result (i).

To get result (ii) put $x = -1$ in (I) ; we have

$$F\left(\frac{1}{x}, \frac{1}{x}, 1, -1\right) = \frac{1}{\pi}\sin\frac{\pi}{x}\int_0^1 t^{\frac{1}{x}-1}(1-t^2)^{-\frac{1}{x}}\,dt$$

$$= \frac{1}{2\pi}\sin\frac{\pi}{x}\int_0^1 z^{\frac{1}{2x}-1}(1-z)^{-\frac{1}{x}}\,dz \quad (z = t^2)$$

$$= \frac{1}{2\pi}\sin\frac{\pi}{x}\,\frac{\Gamma\left(\frac{1}{2x}\right)\Gamma\left(1-\frac{1}{x}\right)}{\Gamma\left(1-\frac{1}{2x}\right)} \quad (x > 1)$$

(iii) We know that if $\alpha + \beta + \frac{1}{2} = \gamma$

$$\{F(\alpha, \beta, \gamma, x)\}^2 = \frac{\Gamma(\gamma)\,\Gamma(2\gamma-1)}{\Gamma(2\alpha)\,\Gamma(2\beta)\,\Gamma(\alpha+\beta)}$$

$$\times \sum_0^\infty \frac{\Gamma(2\alpha+n)\,\Gamma(2\beta+n)\,\Gamma(\alpha+\beta+n)}{n!\,\Gamma(\gamma+n)\,\Gamma(2\gamma-1+n)}\,x^n$$

(*vide* Whittaker, p. 298, Ex. 11).

Hence

$$\{F(\tfrac{1}{4}, \tfrac{1}{4}, 1, x)\}^2 = \left(\frac{1}{\Gamma(\frac{1}{2})}\right)^3 \sum_0^\infty \frac{\Gamma(\frac{1}{2}+n)^3}{n!\,\Gamma(1+n)^2}\,x^n$$

Now $\qquad \Gamma(\frac{1}{2}+n) = \dfrac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,2\,.\,2\,\ldots\,2}\,.\,\Gamma(\frac{1}{2})$

and $\qquad \Gamma(1+n) = n!$

$$\therefore \qquad \{F(\tfrac{1}{4}, \tfrac{1}{4}, 1, x)\}^2 = \sum_0^\infty \left(\frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{2\,.\,4\,\,6\,\ldots\,2n}\right)^3 .\,x^n.$$

To get Ramanujan's result put $x = -1$, we have

$$1 - \frac{1^3}{2^3} + \frac{1^3\,.\,3^3}{2^3\,.\,4^3} - \frac{1^3\,.\,3^3\,.\,5^3}{2^3\,.\,4^3\,.\,6^3} + \ldots\ldots = \{F(\tfrac{1}{4}, \tfrac{1}{4}, 1, -1)\}^2;$$

In (ii) put $x = 4$, and we get

$$F\left(\tfrac{1}{4}, \tfrac{1}{4}, 1, -1\right) = \frac{1}{2\pi} \sin\frac{\pi}{4} \frac{\Gamma(\frac{1}{8})\,\Gamma(\frac{3}{4})}{\Gamma(\frac{7}{8})}.$$

But
$$\frac{1}{\pi}\sin\frac{\pi}{4} = \frac{1}{\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})}.$$

$$\therefore \quad F\left(\tfrac{1}{4}, \tfrac{1}{4}, 1, -1\right) = \frac{\frac{1}{8}\,\Gamma(\frac{1}{8})}{\frac{1}{4}\,\Gamma(\frac{1}{4}).\,\Gamma(\frac{7}{8})} = \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})\,\Gamma(\frac{7}{8})}$$

and hence the result.

Question 1498.

(N. DURAIRAJAN) :—Show that for any point on the nine-points circle of a triangle the following relation holds:

$$\frac{\sin 2\theta_1}{r_1^2} + \frac{\sin 2\theta_2}{r_2^2} + \frac{\sin 2\theta_3}{r_3^2} = 0$$

r_1, r_2, r_3 and $\theta_1, \theta_2, \theta_3$ being the tripolar and triangular co-ordinates of the point.

Solution (i) *by A A. Krishnaswami Ayyangar*
(ii) *by M. Bhimasena Rao.*

(i) Let D, E, F be the mid. pts. of the sides BC, CA, AB respectively, and P any point on the nine-points circle. Denote PD, PE, PF by ρ_1, ρ_2, ρ_3 respectively.

Since D, E, F, P are concyclic, we have by Ptolemy's theorem,

$$\rho_1 \cdot \frac{a}{2} \pm \rho_2 \cdot \frac{b}{2} \pm \rho_3 \cdot \frac{c}{2} = 0,$$

i.e., $$\rho_1 a \pm \rho_2 b \pm \rho_3 c = 0. \qquad \text{... (1)}$$

But $$4\rho_1^2 = r_2^2 + r_3^2 + 2r_2 r_3 \cos\theta_1$$

and $$a^2 = r_2^2 + r_3^2 - 2\, r_2 r_3 \cos\theta_1$$

with similar expressions for ρ_2, ρ_3, b, and c.

Hence (i) reduces to

$$\sqrt{r_2^4 + r_3^4 - 2r_2^2 r_3^2 \cos 2\theta_1} \pm \sqrt{r_3^4 + r_1^4 - 2r_3^2 r_1^2 \cos 2\theta_2}$$
$$\pm \sqrt{r_1^4 + r_2^4 - 2\, r_1^2 r_2^2 \cos 2\theta_3} = 0 \qquad \text{... (2)}$$

To rationalise (2), we observe that if we set three lines OP, OQ, OR of lengths r_1^2, r_2^2, r_3^2 such that

$$P\hat{O}Q = 2\pi - 2\theta_3,\ Q\hat{O}R = 2\pi - 2\theta_1,$$

and hence

$$R\hat{O}P = 2\pi - 2\theta_2,$$

the relation (2) merely implies that P, Q, R are collinear ; so that

$$\Delta\, OPQ \pm \Delta\, OQR \pm \Delta\, ORP = 0.$$

$$\therefore \quad r_1^2 r_2^2 \sin 2\theta_3 + r_2^2 r_3^2 \sin 2\theta_1 + r_3^2 r_1^2 \sin 2\theta_2 = 0,$$

i.e.,

$$\frac{\sin 2\theta_1}{r_1^2} + \frac{\sin 2\theta_2}{r_2^2} + \frac{\sin 2\theta_3}{r_3^2} = 0.$$

NOTE.—It can be easily verified that the usual method of rationalisation of (1) leads to the result

$$(r_1^2 r_2^2 \sin 2\theta_3 + r_2^2 r_3^2 \sin 2\theta_1 + r_3^2 r_1^2 \sin 2\theta_2)^3 = 0.$$

(ii) Consider a triangle LMN whose angles are the supplements of θ_1, θ_2, θ_3 and P a point on the polar circle of LMN. If α, β, γ are the normal co-ordinates of P with respect to LMN,

then $$\alpha^2 \sin 2\theta_1 + \beta^2 \sin 2\theta_2 + \gamma^2 \sin 2\theta_3 = 0. \quad \ldots \quad (1)$$

If ABC be the reciprocal triangle of LMN with respect to a circle having centre at P, the polar circle reciprocates into a parabola with its focus at P and having ABC for a self-polar triangle. By a well-known theorem, P is on the nine-point circle of ABC.

Also $$\alpha\,.\,PA = \beta\,.\,PB = \gamma\,.\,PC,$$

i.e., $$\alpha r_1 = \beta r_2 = \gamma r_3 \quad \ldots \quad (2)$$

and $$\text{angle } BPC = \theta_1, \text{ etc.}$$

From (1) and (2), the required result is at once obtained.

Also solved analytically by V. A. Mahalingam, G. V. Krishnaswamy Aiyangar, B. Ramamurthy, Hukam Chand, and S. V. Minakshisundaram.

Question 1500.

(S. NARAYANA AIYAR, M.A.) :—Establish the following relations :—

(1) $$dn\,\frac{K}{3} = k'\,\frac{1 + k^2\,sn^2\frac{K}{3}\,sn^2\frac{2K}{3}}{1 - k^2\,sn^2\frac{K}{3}\,sn^2\frac{2K}{3}};$$

(2) $$dn\,\frac{K}{2} = k'\,\frac{1 + k\,sn^2\frac{K}{4}\,sn^2\frac{3K}{4}}{1 - k^2\,sn^2\frac{K}{4}\,sn^2\frac{3K}{4}};$$

(3) $$dn\,\frac{3K}{5} = k'\,\frac{1 + k^2\,sn^2\frac{K}{5}\,sn^2\frac{4K}{5}}{1 - k^2\,sn^2\frac{K}{5}\,sn^2\frac{4K}{5}}.$$

Solution by Budhram and S. D. Chowla, Hukam Chand, K. V. Vedantham, N. P. Subramaniam, and Hans R. Gupta.

These formulæ are special cases (n = 3, 4 and 5 respectively) of the following relation :—

$$dn\left(\frac{n-2}{n}K\right) = k'.\frac{1 + k^2\,sn^2\frac{K}{n}\,sn^2\frac{(n-1)}{n}K}{1 - k^2\,sn^2\frac{K}{n}\,.\,sn^2\frac{(n-1)}{n}K} \quad \dots \quad (1)$$

The second member of (1) is equal to

$$k'.\frac{dn^2\frac{K}{n} + k^2\,sn^2\frac{K}{n}\,cn^2\frac{K}{n}}{dn^2\frac{K}{n} - k^2\,sn^2\frac{K}{n}\,cn^2\frac{K}{n}}$$

$$= k'.\frac{1 - k^2\,sn^4\frac{K}{n}}{dn^2\frac{K}{n} - k^2\,sn^2\frac{K}{n}\,cn^2\frac{K}{n}} = \frac{k'}{dn\,\frac{2K}{n}}$$

$$= dn\left(\frac{n-2}{n}K\right).$$

Hence (1) is proved for all real values of n.

Question 1507.

(N. Durairajan) :—Given four points in a plane, show that the conics which have their centres at any one of the four points, and have the triangle formed by the other three for a self-polar triangle, are similar conics.

Solution (1) *by B. Ramamurthi*; (2) *by A. Ranganatha Rao*; (3) *by G. V. Krishnaswami, K. V. Vedantham and B. Ramamurthi.*

(1) If A be the centre of a conic and B the pole of CD with respect to it, we know that AB, CD are conjugate directions, *i.e.*, parallel to a pair of conjugate diameters. Hence it follows that *for all the four conics*, the pairs of opposite sides of the quadrangle ABCD are conjugate directions. Hence the three pairs of opposite sides meet the line at infinity in pairs of points belonging to an involution whose double points are the points at infinity on each of the conics. Hence all the conics have the same points at infinity, *i.e.*, they are similar and similarly situated.

(2) Let A, B, C, D be the four points. Let ϕ_A be the conic which has A for centre and BCD for a self-polar triangle. Let two conjugate diameters of ϕ_A meet the line at infinity in P and Q. Then the two triangles APQ and BCD are both self-polar for the conic ϕ_A. Hence a conic passes through the six vertices. If P and Q coincide, this conic is a parabola through A, B, C, D. But two conjugate diameters of ϕ_A cannot coincide unless they do so with an asymptote of ϕ_A. So the two asymptotes of ϕ_A are parallel to the axes of the two parabolas through A, B, C, D. Thus the asymptotes of each of the four conics ϕ_A, ϕ_B, ϕ_C and ϕ_D are parallel to the same pair of straight lines. Therefore the four conics in question are similar.

(3) Let the given points A, B, C, D be projected orthogonally into an ortho-centric tetrad, A′, B′, C′, D′. Then the four given conics project into the polar circles of the triangles of this tetrad. Henc the original conics are similar and similarly situated.

Also solved analytically by Hukam Chand, M. V. Seshadri and K. V. Vedantham.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1592. (PROF. S. MUKHOPADHYAYA) :—C denotes an ellipse, parabola or single branch of a hyperbola. EF is a chord of length k equal to or less than l, the latus rectum. The tangents at E and F meet at T. The part of C which lies on the T-side of EF is S. The remainder of C is S$'$. $\theta = \sin^{-1}\frac{k}{l}$. P is a point of S and P$'$ is a point of S$'$. Show that

(i) angles TEF and TFE are each less than θ;

(ii) angle EPF is greater than $\pi - \theta$;

(iii) angle EP$'$F is less than θ.

1593. (B. B. BAGI) :—O is any point in the plane of a triangle ABC. BC, CA, AB, OA, OB, OC are of lengths a, b, c, a', b', c' and they make angles α, β, γ and α', β', γ' with a straight line respectively. Prove that there exists a triangle whose sides are proportional to aa', bb', cc' and whose exterior angles are λ, μ, ν where λ, μ and ν are given by

$$\lambda = \gamma + \gamma' - \beta - \beta', \qquad \mu = \alpha + \alpha' - \gamma - \gamma'$$

and

$$\nu = \beta + \beta' - \alpha - \alpha'$$

correct to a multiple of 2π.

Give a geometrical construction for the triangle.

1594. (V. RAMASWAMI AIYAR, M.A.) :—"Given any two conjugate semi-diameters OP, OQ of an ellipse, to find the magnitude and

direction of its axes." Mannheim's beautiful construction for the magnitude and direction of the axes is given in p. 210 of Casey's *Conics* (2nd edition). Do you see any defect in the proof given of the construction? Complete the proof.

(*b*) In connection with the foregoing problem and construction, show also how the semi-diameter of the ellipse, of any prescribed direction, can be readily constructed.

1595. (V. Ramaswami Aiyar, M A.):—"CB is a fixed diameter of a given circle; and through a fixed point A in CB draw any chord DE of the circle; join CD, and on CD produced if necessary, take CF = AE: the locus of the point F is a hyperbola." [Attributed to Hamilton, see Casey's *Conics* (2nd edition), p 253]. Is there an error in this; and if so, is it to be attributed to Hamilton, or to Casey's *quotation* from him?

Unsolved Questions for which solutions are invited.

381. (V. Ramaswami Aiyar, M.A.):—O is the circum-centre of a triangle ABC, PQ is any diameter of an inscribed conic. If D be a point in BC such that the angles AOD and POQ have the same bisectors, show that the feet of the perpendiculars from O on the sides of the quadrilateral APDQ lie on a circle touching the nine-points circle.

420. (K. J. Sanjana M.A.):—In the plane of three fixed points A, B, C, a fixed straight line is drawn through A, and a variable circle touches it at that point. Prove that the locus of the intersection of the line joining B to the centre of the circle and the polar of C with regard to the circle is a conic. (Geometrical Proof preferred). Show that Q. 390 is a particular case of this theorem.

If BM, CN are perpendicular to the fixed straight line, examine the locus when AM.AN = — BM.CN.

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of April and May 1931.

1 Acta Mathematica, **56**, 4.
2 American Journal of Mathematics, **53**, 2.
3 American Mathematical Monthly, **38**, 3, 4.
4 Annals of Mathematics, **32**, 2.
5 Annales de l'Ecole Normale Superieure Jan., Feb., March and April 1931.
6 Astrophysical Journal, **72**, 1, 2, 3.
7 Bulletin of the American Mathematical Society, **37**, 1, 2, 3.
8 Bulletin of the Calcutta Mathematical Society, **22**, 4.
9 Bulletin des Sciences Mathematiques, **55**, Feb and Mar. 1931.
10 Crelle's Journal, **164**, 2, 3.
11 Indian Physico-Mathematical Journal, **11**, 1.
12 Mathematical Gazette, **15**, 212, 213 (3 copies each).
13 Mathematische Annalen, **104**, 3, 4.
14 Monthly Notices of the Royal Astronomical Society, **91**, 4.
15 Philosophical Magazine, **11**, 72.
16 Philosophical Transactions of the Royal Society, **230**, 75—101, 103—147.
17 Proceedings of the Royal Society of London, **130**, 815, **131**, 816. 817.
18 Proceedings of the London Mathematical Society, **32**, 1, 2, 3 (3 copies each).
19 Quarterly Journal of Mathematics, **2**, 5.
20 Rendiconti del Seminario Mathematico e Fisico di Miland, **4**.
21 Rendiconti del circolo Matematico de Palermo, **55**, 1.
22 Revue Semestrlle des Publications Mathematiques, **35**.
23 Studia Matematica, **2**.
24 Rendiconti del Seminario Mathematico, 1, 3, 4.
25 Revista Matematica Hispano American, Madrid, **4**, 1, 2.
26 Transactions of the Royal Society of South Africa, **19**, 3.

Books.

1 Calendar of the University of Madras, 1930—1931.
2 Encyklopadie der Mathematischen Wissenschaften Band III$_1$ Heft 10, Band VI$_{2B}$ Heft 3.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues

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